*7-Apr finished last problem @ end

(continuing thought regarding example from end of Lecture 15)

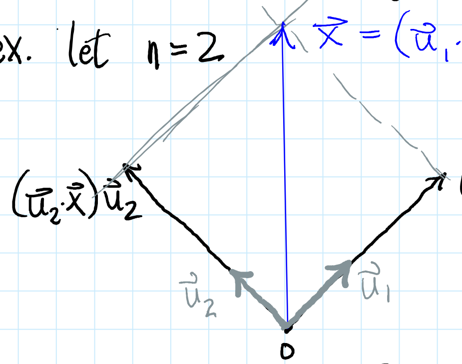
§5.1 cont'd

then, it can be concluded:

thm: given orthonormal basis $\vec{u}_1, \dots, \vec{u}_n$ of $\mathbb{R}^n$
$\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_n \cdot \vec{x})\vec{u}_n \quad \forall \vec{x} \in \mathbb{R}^n$

i.e., if $\vec{x}$ is projected onto all lines spanned by basis vectors $\vec{u}_i$ and resultant vectors are added, result is $\vec{x}$

ex. let $n=2$



How is this applied?

when $\exists$ basis $\vec{v}_1, \dots, \vec{v}_n$ of $\mathbb{R}^n$, any $\vec{x} \in \mathbb{R}^n$ can be uniquely expressed as a LC of $\vec{v}_i$: $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

then, to find coordinates, c_i, need to solve linear system, which could be cumbersome

however, if basis is <u>orthonormal</u>, can easily determine c_i: $\boxed{c_i = \vec{u}_i \cdot \vec{x}}$

ex. given orthogonal projection $T(\vec{x}) = \text{proj}_V \vec{x}$ onto subspace V of $\mathbb{R}^n$

a. describe $\text{im}(T)$ ← subspace of V

b. describe $\ker(T)$ ← consist of all vectors $\vec{x} \in \mathbb{R}^n$ s.t. $T(\vec{x}) = \text{proj}_V \vec{x} = \vec{x}_{\parallel} = \underline{\underline{0}}$

know $\vec{x} = \cancelto{0}{\vec{x}_{\parallel}} + \vec{x}_{\perp}$

$\Rightarrow \vec{x} = \vec{x}_{\perp}$

i.e., $\ker T = \vec{x} \in \mathbb{R}^n$ that are orthogonal to V

def'n: given subspace V on $\mathbb{R}^n$

$V_\perp$ of V is called <u>orthogonal complement</u> and is set of $\vec{x} \in \mathbb{R}^n$ that are orthogonal to all vectors in V

also, $V_\perp$ is the kernel of orthogonal projection onto V

<u>Properties of Orthogonal Complement</u>

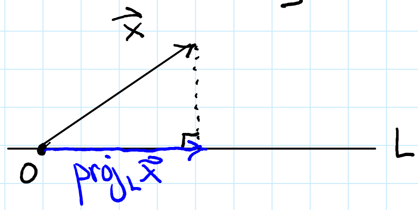
given subspace V on $\mathbb{R}^n$:

- contain $\vec{0}$
- closed under LC

- orthogonal complement $V_\perp$ of V is a subspace of $\mathbb{R}^n$
- $V \cap V_\perp = \{\vec{0}\}$
- $\dim(V) + \dim(V_\perp) = n$
- $(V_\perp)_\perp = V$

<u>Expanding Pythagorean Theorem</u>

given line $L \in \mathbb{R}^3$, $\vec{v} \in \mathbb{R}^3$ — what is the relationship between the <u>lengths</u> of $\vec{x}$ and $\text{proj}_L \vec{x}$?



$$\|\text{proj}_L \vec{x}\| \leq \|\vec{x}\|$$

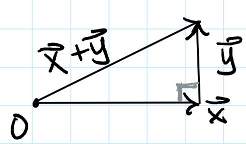
statement is an equality iff $\vec{x}$ is on L

Q: Does Pythagorean Theorem hold in $\mathbb{R}^n$?

given $\vec{x}, \vec{y}$ in $\mathbb{R}^n$

then $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ holds iff $\vec{x}, \vec{y}$ are <u>orthogonal</u>

when $\vec{x} \cdot \vec{y} = 0$



Proof:

$$\begin{aligned}\|\vec{x}+\vec{y}\|^2 &= (\vec{x}+\vec{y}) \cdot (\vec{x}+\vec{y}) \\ &= \vec{x}\cdot\vec{x} + 2(\vec{x}\cdot\vec{y}) + \vec{y}\cdot\vec{y} \\ &= \|\vec{x}\|^2 + 2\cancel{(\vec{x}\cdot\vec{y})} + \|\vec{y}\|^2\end{aligned}$$

only holds when $\vec{x}\cdot\vec{y} = 0$

then $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ $\square$

thm: given subspace V on $\mathbb{R}^n$

then $\|x+y\| = \|x\| \quad + \|y\|$ $\square$

thm: given subspace V on $\mathbb{R}^n$
$\vec{x} \in \mathbb{R}^n$

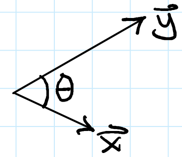
then $\|\text{proj}_V \vec{x}\| \leq \|\vec{x}\|$ and is an equality iff $\vec{x} \in V$

<u>Cauchy-Schwarz Inequality</u>

if $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^n$
then $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \, \|\vec{y}\|$ and is an equality iff $\vec{x}$ and $\vec{y}$ are parallel

given 2 non-zero vectors $\vec{x}, \vec{y} \in \mathbb{R}^3$

then dot product can be represented as:



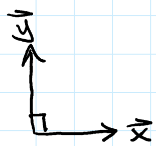
$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos\theta$ ← can be proved w/ Law of Cosines

solve for θ:

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|} \longrightarrow \boxed{\theta = \arccos \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}}$$

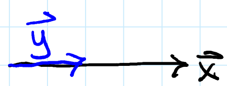
where $0 \leq \theta \leq \pi$ by def'n of arccos$x$

recall: if $\vec{x} \cdot \vec{y} = 0$



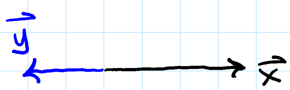
$\perp$

if $\vec{x} \cdot \vec{y} = 1$



$\parallel$ same direction

if $\vec{x} \cdot \vec{y} = -1$



$\parallel$ opposite direction

$$\therefore \; -1 \leq \vec{x} \cdot \vec{y} \leq 1 \quad \Rightarrow \quad \left| \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|} \right| = \frac{|\vec{x} \cdot \vec{y}|}{\|\vec{x}\| \, \|\vec{y}\|} \leq 1$$

↓ confirms Cauchy-Schwarz Inequality:

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \, \|\vec{y}\|$$

ex. Find angle between $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

$\vec{x} \cdot \vec{y} = 1$

$$\text{use } \theta = \arccos \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|}$$

$$= \arccos \frac{1}{1 \cdot 2}$$

$$= \arccos \frac{1}{2} = \boxed{\frac{\pi}{3}} \quad \left(\text{since } \cos\frac{\pi}{3} = \frac{1}{2}\right)$$

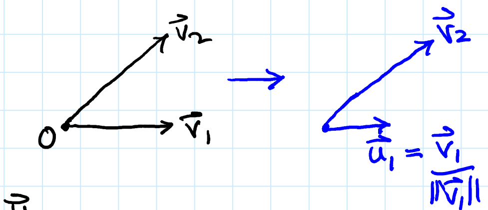
§ 5.2 continuing with orthogonal projection onto subspace V of $\mathbb{R}^n$

$\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$ where $\vec{u}_1, \dots, \vec{u}_m$ is an orthonormal basis of V

now develop a method to construct an orthonormal basis

1st, if V is a plane with basis $\vec{v}_1, \vec{v}_2$

construct $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

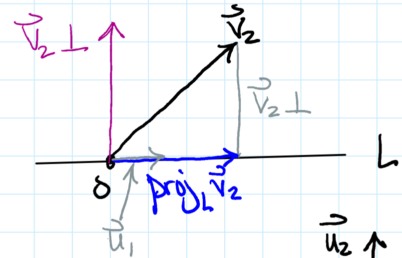


then find a vector in V orthogonal to $\vec{u}_1$

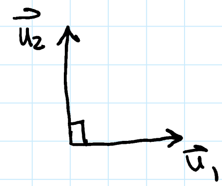
↑ for now, not necessarily a unit vector

$$\vec{v}_2 = \vec{v}_{2\parallel} + \vec{v}_{2\perp}$$
$$v_{2\perp} = \vec{v}_2 - \vec{v}_{2\parallel}$$
$$= \vec{v}_2 - \text{proj}_L \vec{v}_2$$
$$\vec{v}_{2\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$

△ confirmed



last, divide $\vec{v}_{2\perp}$ by its length: $\vec{u}_{2\perp} = \dfrac{\vec{v}_2 \perp}{\|\vec{v}_2 \perp\|}$



ex. find orthonormal basis $\vec{u}_1, \vec{u}_2$ of subspace V

given $V = \text{span}\left(\underbrace{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix}1\\9\\9\\1\end{bmatrix}}_{\vec{v}_2}\right)$ of $\mathbb{R}^4$ with basis $\vec{v}_1, \vec{v}_2$

$\|\vec{v}_1\| = \sqrt{4} = 2$

① determine $\vec{u}_1 = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$

② determine $\vec{v}_2 \perp = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$

$$= \begin{bmatrix}1\\9\\9\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} \cdot \begin{bmatrix}1\\9\\9\\1\end{bmatrix} \vec{u}_1$$

$$= \begin{bmatrix}1\\9\\9\\1\end{bmatrix} - \cancel{\frac{1+9+9+1}{2}}\,10\ \vec{u}_1$$

$$= \begin{bmatrix}1\\9\\9\\\end{bmatrix} - 10\begin{bmatrix}1/2\\1/2\\1/2\\\end{bmatrix} = \begin{bmatrix}1\\9\\9\\\end{bmatrix} - \begin{bmatrix}5\\5\\5\\\end{bmatrix} = \begin{bmatrix}-4\\4\\4\\\end{bmatrix} = \vec{v}_2 \perp$$

$$= \begin{bmatrix} 9 \\ 9 \\ 1 \end{bmatrix} - 10 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \vec{v}_2 \perp$$

$$\|\vec{v}_2 \perp\| = 8$$

$$\therefore \vec{u}_2 = \frac{\vec{v}_2 \perp}{\|\vec{v}_2 \perp\|} = \frac{1}{8} \begin{bmatrix} -4 \\ 4 \\ 4 \\ -4 \end{bmatrix} \rightarrow \vec{u}_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$\|\vec{v}_2 \perp\| = \sqrt{(-4)^2 + 4^2 + (4)^2 + (-4)^2} = \sqrt{64} = 8$$

appended 7 Apr

Conclude: orthonormal basis $\mathcal{U}$ of V is:

$$\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$